UNSW SYDNEY | Australia's Global University

School of Mathematics and Statistics

HSC Tips and Tricks: Open Day 2019

Hosted by the UNSW Mathematics Society

TOPICS COVERED INCLUDE

Approaching HSC Maths
General Exam Tips
Complex Number Square Roots
Polynomial Long Division
Partial Fractions
Advanced Combinatorics
Harder Inequalities

TWO SESSIONS

- 10:50am - 11:20am
  Mathews Theatre A
  Map ref: D23

- 2:10pm - 2:40pm (repeat)
  Mathews Theatre B
  Map ref: D23

MathSoc

SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES

HSC TIPS AND TRICKS UNSW Open Day 2019

This handout was written by Peter Ayre, Johann Blanco, Gary Liang, Vishaal Nathan, Varun Nayyar, Brendan Trinh, Georgia Tsambos and Aaron Hassan. It was updated by Abdellah Islam and Jeffrey Yang. The suggestions made are those of the authors, and are not officially endorsed by the School of Mathematics and Statistics or the Faculty of Science.

If you choose to come to UNSW (which we hope you do!), please consider supporting us by joining our society! Happy studying and good luck with the HSC! ☺

Many thanks to the School of Mathematics and Statistics and the Faculty of Science for helping us organise this event (and printing this handout).

Contents

1 Approaching HSC Mathematics

Doing well in HSC mathematics comes down to two things:

Understanding means you know **why**. If someone asks you why you do something in maths, you need to be able to justify it. Mathematics is *not* about rote learning all the types of questions. Inevitably, there will be questions you have not seen before. Here are some ways to help you understand:

→**Always question 'Why?'** Don't just accept what you are being told. (However, keep in mind that sometimes, certain things are true by definition.)

→**Go back to the fundamentals.** The number one reason why you won't understand a concept is not knowing what underlies it. For example, differentiation won't be fully understood if gradients and limits are not understood properly.

→**Teach others.** This is by far the best test to see whether you truly understand something or not. As Einstein said, "If you can't explain it to a six year old, you don't understand it yourself."

→**Always ask yourself the "dumb" questions**. Sometimes, the ideas or principles underlying a rather obvious result are not obvious at all. Never be afraid to ask yourself and others the "dumb" question as the answers to these questions can lead to surprising conclusions, but more often will simply tell you why the principles hold in the first place, which is well worth knowing.

→**Perfecting** actually means getting all the marks from understanding your content. This involves practising a lot so that any careless mistakes are eliminated. A great example is the index laws: many students know them, but not to a point where they are second nature, and as a result, marks are needlessly lost. Practice to an extent where you hardly have to think about what you are doing.

→**Do every question.** Homework is extremely important. The truth is, you might understand a concept, but to get all the marks from it, you need to know how to do the questions essentially without even thinking. Make sure you do enough practice until each question becomes second nature.

→**Do past papers.** Past papers are the most important way to do well in HSC mathematics. Past papers expose you to a variety of different question types that could be asked during an exam. When doing past papers, do them in exam conditions so that you get used to the time limits and the exam pressure, so when the time does come, you won't be as nervous!

→**Work on your basic algebra.** Why a lot of students get questions wrong is a lack of skill in basic algebra. Most students don't know their index laws, their expansion methods and factorisation to a point where they don't need to think about it. If there is one thing to work on in Year 11 and below, it is basic algebra, because it makes everything else so much easier.

2 General Exam Tips

Use reading time well

Make sure that you thoroughly read all the questions during reading time, especially the ones at the back. One of the most common mistakes made in an exam is not reading the question properly. Slow yourself down and make sure you are reading every word.

Also, don't bother doing multiple choice during reading time (unless it is very straightforward) – there's a higher chance of you getting it wrong when you are doing it in your head, and reading the later questions is a better use of time.

Attempt every question

Don't go into an exam with the mentality "I'm not going to bother looking at the last question." Consider 2010 HSC, Question 8, part (j) for example – a simple 1 mark question, just taking the limit to infinity of the expression given in (i). Remember, you can assume results of earlier parts of a question to do later parts even if you were unable to prove them.

(HSC 2010, 8 (i)) Use part (e) to deduce that

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6}.$$

(HSC 2010, 8 (j)) What is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}?$$

Induction conclusion

While doing a proof by mathematical induction, there's no need to write the paragraph your school probably asks you to write out. In reality, mathematical induction is an axiom of mathematics and it is simply enough to write something like "Hence, by mathematical induction, the statement is true for all integers $n \geq 1$."

However, unfortunately many teachers mistakenly force their students to write it all out and even deduct marks for not doing so, so make sure you confirm with your teacher what they expect of you for induction questions during your internal exams. For the external HSC exam, there is definitely no need to write it all out.

Calculator Tricks

For Extension 2 students, note that certain calculators like the Casio fx-100AU can do simple complex number operations including adding, subtraction, dividing and multiplication, along with conversion from Cartesian to polar form.

Saving Time in Multiple Choice

In the HSC exam, there are 10 multiple choice questions. As these are each worth only 1 mark, you don't want to spend excessively long on these, or do more work than you need to. All you need is the correct answer, and then you can move on. You want to effectively be able to spot the right answer with minimal work and time. Here we describe some tricks that can let you spot the correct answer quickly, often with almost no working required, and usually much faster than by solving the question thoroughly by conventional methods.

Checking units

Sometimes the answer to multiple choice questions (particularly Mechanics questions) can be done quickly by consideration of units. Using the method of "checking units", we see what the physical units of the answer should be, and check what the units of the given multiple choice options are. Often, only one option actually has the correct units, in which case it must be the answer. This gives us a quick way to get the answer without having to do any hard calculations or working out. Even if multiple options have the correct units, we can at least eliminate options that do not have the right units.

As an example, consider the following multiple choice question.

Example 1: A particle is undergoing Simple Harmonic Motion. Its velocity v satisfies the equation $v^2 = \alpha^2 - \beta^2 x^2$, where x is its displacement from the origin and α, β are positive constants. What is the period of the motion?

(A) α (B) β (C) $2\pi/\alpha$ (D) $2\pi/\beta$

Solution 1: We do this by consideration of units. Using SI units, the units for a period is s (since "period" is a "time" quantity). From the equation $v^2 = \alpha^2 - \beta^2 x^2$, we know the LHS has units $\mathrm{m}^2\ \mathrm{s}^{-2}$ (because the units of v are $\mathrm{m\ s}^{-1}$). Therefore, the RHS must have units of $\mathrm{m}^2\ \mathrm{s}^{-2}$. Hence each of the terms in the RHS have these as units, since we can only add or subtract two quantities that have the same units. Hence α must have the same units as v ($\mathrm{m\ s}^{-1}$) and $\beta^2 x^2$ must have units of $\mathrm{m}^2\ \mathrm{s}^{-2}$, which implies that β^2 has units of s^{-2}, i.e. β has units of s^{-1}. Therefore, option (A) has units of $\mathrm{m\ s}^{-1}$, option (B) has units of s^{-1}, option (C) has units of $(\mathrm{m\ s}^{-1})^{-1} = \mathrm{m}^{-1}\mathrm{s}$, and option (D) has units of s. As we can see, only option (D) has the correct units for a period, so the answer is (D).

Example 2: A particle of mass m undergoes uniform circular motion in a conical pendulum of vertical height h, where the radius of rotation is r, semi-vertical angle is α, tension in the string is T, and angular velocity is ω. Which of the following is a correct expression for ω?

(A) $\sqrt{\frac{h}{r}}$ (B) $\sqrt{\frac{r}{h}}$ (C) $\sqrt{\frac{g}{h}}$ (D) $\sqrt{\frac{h}{g}}$

Solution 2: Using the method of checking units, the answer is (C). We know that (using SI units), the units of ω are radians per second, which can just be written as s^{-1} because radians, being a measure of angle, are measuring a unitless quantity (angles are dimensionless). The options (A) and (B) do not contain "seconds" anywhere as they involve only lengths, so those options cannot be correct. The options (C) and (D) have units that are reciprocals of each other, because options (C) and (D) are reciprocals of each other. The units of g are $\mathrm{m\ s^{-2}}$ (usual units of an acceleration quantity, which gravity is) and the units of h are m (units for a length quantity). Hence the units of $\frac{g}{h}$ will be $\frac{\mathrm{m\ s^{-2}}}{\mathrm{m}} = \mathrm{s^{-2}}$, so the units of $\sqrt{\frac{g}{h}}$ are $\sqrt{\mathrm{s^{-2}}} = \mathrm{s^{-1}}$. Hence option (C) has the correct units for angular velocity, so <u>(C) is the answer.</u> (As remarked earlier, the units of (D) are reciprocal to that of (C), so (D) has units of s, so cannot be right for angular velocity.)

Testing easy special cases

Sometimes the answer to an otherwise time-consuming or challenging multiple choice question can be deduced by answering the question in a simple special case for which we can easily find the answer. We will demonstrate on the following example.

Example 3: An urn contains n marbles, and each marble is either red or blue. It is known that k of the marbles in the urn are red. Two marbles are randomly selected from the urn (without replacement). *Given* that at least one of the chosen marbles is red, what is the probability that both marbles chosen are red?

(A) $\frac{k-1}{2n-k-1}$ (B) $\frac{k-1}{2n+k+1}$ (C) $\frac{k-1}{2n+k-1}$ (D) $\frac{k-1}{2n-k+1}$

Solution 3: Consider the simple case where $k = n$. In this case, *all* the marbles in the urn are red, so that the probability in question is clearly just 1 (since we are guaranteed to have chosen both marbles as red in this case). Therefore, whatever the answer is, it must be equal to 1 in the case of $k = n$. If we substitute $k = n$ into each of the formulas in options (A)-(D), we find that only option (A) simplifies to 1! Therefore, the <u>answer is simply (A)</u>.

This demonstrates how we can get the answer to an otherwise potentially messy multiple choice question with minimal work by just testing some easy special cases for which we know the answer. It is a good exercise for your probability practice to derive the answer to this though as if it were a written response question (so as if you were not given any options for the answer), but for the purposes of a multiple choice question, it often pays to first look at easy cases.

Finally, the tricks described in this section help save time on multiple choice, *but they are good general principles to keep in mind even for written response*. So whenever you have mechanics-based questions for example, it is a good idea to check the units of your final answer to make sure it makes sense. Similarly, if you ever have to derive a formula for some general problem involving variables (like n and k in the probability question), it is a good idea to test your formula with simple special cases for which you know what the answer should be, as this provides a check on your general formula.

3 Complex Number Square Roots

Finding square roots of complex numbers can be tedious. For instance, let's try to find the square root of $3 + 4i$:

> Let $3 + 4i = (x + yi)^2$ for some real numbers x and y. We then expand and equate real and imaginary parts as such:
>
> $$3 + 4i = x^2 - y^2 + 2xyi.$$
>
> Hence,
>
> $$3 = x^2 - y^2 \text{ and } 4 = 2xy.$$
>
> We must solve these simultaneously, and most would use the second equation to get that $y = \frac{2}{x}$, and then:
>
> $$\begin{aligned} 3 &= x^2 - \left(\frac{2}{x}\right)^2 \\ 3x^2 &= x^4 - 4 \\ x^4 - 3x^2 - 4 &= 0 \\ (x^2 - 4)(x^2 + 1) &= 0 \\ x &= \pm 2. \end{aligned}$$
>
> Substitution yields that $y = \pm 1$, so our solutions are $2 + i$ and $-2 - i$. ■

This process, however, can be shortened.

Using properties of the modulus

When solving for square roots as above, we're solving something of the form $z^2 = w$. In our example above, $w = 3 + 4i$, for instance, we can use properties of the modulus to simplify the type of equation we need to solve. Observe that:

$$\begin{aligned} z^2 &= w \\ |z^2| &= |w| \\ |z|^2 &= |w|. \end{aligned}$$

So, in our previous example, this would resolve down to:

$$\begin{aligned} |x + yi|^2 &= |3 + 4i| \\ x^2 + y^2 &= 5. \end{aligned}$$

Now we have three equations, isn't that more complicated? No. Our three equations are:

$$x^2 - y^2 = 3$$
$$x^2 + y^2 = 5$$
$$2xy = 4$$

If we add the first and second together, we get that $2x^2 = 8$, and hence $x = \pm 2$. Substitute this into the third equation to get our full answer, and also means we have less of a chance of getting the signs wrong.

Completing the Square

With *some* complex numbers, we can complete the square and find square roots in a couple of lines. For example, using our original example:

$$3 + 4i = 4 + 4i + i^2$$
$$3 + 4i = (2 + i)^2.$$

And therefore, our square roots are $\pm(2 + i)$.

You can do this with many other complex numbers too:

$$\pm 2i = 1 \pm 2i + i^2 = (1 \pm i)^2$$
$$8 \pm 6i = 9 \pm 6i + i^2 = (3 \pm i)^2$$
$$15 \pm 8i = 16 \pm 8i + i^2 = (4 \pm i)^2$$

And using some neat tricks, we can do the same method on complex numbers that are derived from these. For example,

$$i = \frac{2i}{2} = \frac{(1 + i)^2}{2}$$

And so, the roots are $\pm \frac{1}{\sqrt{2}}(1 + i)$. Moreover, we can do the same trick on $4 + 3i$. See that

$$4 + 3i = i(3 - 4i) = i(4 - 4i + i^2) = i(2 - i)^2.$$

We know the root of i from above, so working through it, you should find that the roots are $\pm \frac{1}{\sqrt{2}}(3 + i)$.

Now, this seems a bit out of left field – how do I know when to use it? The trick here is practice, not only practice of the method but also recognising when you can use it. It doesn't work all the time!

Consider this method an additional tool in your "mathematical toolbox" of how to solve certain problems.

Modulus-argument form

Taking square roots of some numbers can be more easily done if the argument is easily found. For example, suppose we needed to find the square roots of $1+\sqrt{3}i$. The trick in this case is to add $2k\pi$ to the argument:

$$1+\sqrt{3}i = 2\left(\cos\left(\frac{\pi}{3}+2k\pi\right)+i\,sin\left(\frac{\pi}{3}+2k\pi\right)\right).$$

Now we use De Moivre's theorem to see that:

$$\sqrt{2}\left(\cos\left(\frac{\pi}{3}+2k\pi\right)+i\,sin\left(\frac{\pi}{3}+2k\pi\right)\right)^{\frac{1}{2}} = \sqrt{2}\left(\cos\left(\frac{\pi}{6}+k\pi\right)+i\,sin\left(\frac{\pi}{6}+k\pi\right)\right)$$
$$= \sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\,sin\left(\frac{\pi}{6}\right)\right), \sqrt{2}\left(\cos\left(-\frac{5\pi}{6}\right)+i\,sin\left(-\frac{5\pi}{6}\right)\right)$$
$$= \sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right), \sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$$

by substituting $k=-1,0$.

What this really boils down to is that:

$$(cos(\theta)+i\,sin(\theta))^{\frac{1}{2}} = \cos\left(\frac{\theta}{2}+k\pi\right)+i\,sin\left(\frac{\theta}{2}+k\pi\right) \qquad (k=-1,0)$$
$$= \cos\left(\frac{\theta}{2}\right)+i\,sin\left(\frac{\theta}{2}\right),\ \cos\left(\frac{\theta}{2}-\pi\right)+i\,sin\left(\frac{\theta}{2}-\pi\right).$$

4 Avoiding Polynomial Long Division

Sometimes, there are rational functions that you might use long division on to simplify, such as

$$\frac{x^3}{x-2} \quad \text{or} \quad \frac{x^2-1}{x^2+4}.$$

When working with polynomials in mathematics, you want to avoid long division as much as possible. It is time consuming and prone to mistakes. So how can we avoid it?

Remainder Theorem: This seems obvious, but many students forget this. When you are dividing by a linear factor, you should use the remainder theorem to find remainders. That is, when you divide $P(x)$ by $x-\alpha$, the remainder is $P(\alpha)$.

Algebraic Manipulation: The trick with simplifying fractions is to add and subtract terms so that we get cancellation. Let's go through some examples:

Example 1: Evaluate

$$\int \frac{x^2-1}{x^2+4}\,dx.$$

For those of you who haven't learned integration yet, consider instead just trying to graph the fraction.

The trick here is to add and then subtract something to the numerator so that terms cancel out, like so:

$$\frac{x^2-1}{x^2+4} = \frac{x^2+4-5}{x^2+4} = \frac{x^2+4}{x^2+4} - \frac{5}{x^2+4}$$

Thus, we can simplify and integrate (or graph). The integration becomes simple:

$$\int 1 - \frac{5}{x^2+4}\,dx = x - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 2: What is the quotient when x^3 is divided by $x-2$?

We are looking at the fraction

$$\frac{x^3}{x-2}.$$

Again, we need to add/subtract something to the top and bottom to make it cancel. But what? As there is an x^3, let us try an approach using the difference of two cubes:

$$\frac{x^3-8+8}{x-2} = \frac{x^3-8}{x-2} + \frac{8}{x-2} = \frac{(x-2)(x^2+2x+4)}{x-2} + \frac{8}{x-2}$$

And this simplifies to

$$x^2 + 2x + 4 + \frac{8}{x-2}.$$

So the quotient is $x^2 + 2x + 4$.

Example 3: Here is a harder one with no explanation. Try and justify each step.

$$\begin{aligned}\frac{3x^3 - 5x^2 + x + 1}{x^2 - 2x + 3} &= \frac{(3x^3 - 6x^2 + 9x) + x^2 - 8x + 1}{x^2 - 2x + 3}\\ &= \frac{3x(x^2 - 2x + 3)}{x^2 - 2x + 3} + \frac{(x^2 - 2x + 3) - 6x - 2}{x^2 - 2x + 3}\\ &= 3x + \frac{(x^2 - 2x + 3)}{x^2 - 2x + 3} + \frac{-6x - 2}{x^2 - 2x + 3}\\ &= 3x + 1 + \frac{-6x - 2}{x^2 - 2x + 3}\end{aligned}$$

The method of inspection: If you have a known factor, say from the remainder theorem, you can simply inspect the quotient out of a polynomial instead of going through the whole long division process. For example, given $27x^3 - 168x^2 - 143x - 28$ and the fact that $x = \frac{1}{3}$ is a root (or $3x + 1$ is a factor), we know that the quotient must be a quadratic (of degree two) and of the form $Ax^2 + Bx + C$. That is:

$$(3x + 1)(Ax^2 + Bx + C) \equiv 27x^3 - 168x^2 - 143x - 28.$$

This sort of problem may seem familiar.

If not, we look at coefficients of x^3, x^2, x and the constant term. The constant term, -28, must come from some product of terms on the LHS, for instance. And the coefficient for x^3, which is 27, must also come from some product. It is left to the reader to see that making equations for these, we have that:

$$\begin{aligned}3A &= 27\\ \Rightarrow A &= 9\end{aligned} \qquad\qquad C = -28.$$

And for B, we realise that it is a sum of terms involving coefficients of x or x^2. As we already know what A is, we can form a simple equation for B in two ways, and we do both for completeness:

$$3x \times (Bx) + Ax^2 = -168x^2$$

So:

$$\begin{aligned}3B + A &= -168\\ 3B &= -177\\ B &= -59\end{aligned}$$

$$3x \times C + Bx = -143x$$

So:

$$\begin{aligned}3C + B &= -143\\ B &= -143 + 84\\ B &= -59\end{aligned}$$

This eventually yields:

$$(3x+1)(9x^2-59x-28) = 27x^3-168x^2-143x-28.$$

For higher order division, we do this iteratively (left to right, or right to left) and with practice, you can do the division mentally, which saves time. This is most efficient when dividing cubics or quartics, which is the hardest you'll deal with in the HSC.

Remember this method only applies when the divisor is a factor of the original polynomial.

Try yourself!

1. $\dfrac{x}{x-2}$
2. $\dfrac{3x+4}{x-1}$
3. $\dfrac{x^2}{x^2+1}$
4. $\dfrac{x^3-1}{x+1}$
5. $\dfrac{x^3-2x^2-2x+2}{x-1}$
6. $\dfrac{27x^3-168x^2-143x-28}{9x+4}$, check remainder $=0$
7. $\dfrac{6x^4-11x^3-55x^2+67x+105}{3x-7}$, check remainder $=0$

5 Quick Partial Fractions

What partial fraction decomposition involves is decomposing a rational expression into simpler rational expressions. For example:

$$\frac{3x+11}{(x-3)(x+2)} = \frac{4}{x-3} - \frac{1}{x+2}.$$

This is useful for integration. Now going from right to left is easy – all you need to do is put it under the same denominator. However, going from left to right is considerably more difficult and time consuming.

Here is the typical method of partial fraction decomposition.

Say we have a fraction that want to decompose:

$$\frac{3x+11}{(x-3)(x+2)}.$$

We know we want to get it into the form

$$\frac{A}{x-3} + \frac{B}{x+2}.$$

So we write

$$\frac{3x+11}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2}.$$

Our aim is to find A and B. One important thing to note is that this identity is true for all values of x. We multiply both sides by $(x-3)(x+2)$ to make things easier.

$$3x + 11 \equiv A(x+2) + B(x-3).$$

Now we can do it by substituting in certain values of x. Let's choose $x = -2$ and $x = 3$.

$$5 = -5B \Longrightarrow B = -1$$
$$20 = 5A \Longrightarrow A = 4$$

Certain types of partial fractions can be solved with a lot less effort, specifically fractions where the denominator is a monic quadratic and the numerator is a constant.

Example 1: Reduce $\frac{3}{x^2-4}$ to the form $\frac{A}{x-2} + \frac{B}{x+2}$.

Solution 1: One thing to remember about fractions where the denominator is a monic quadratic and the numerator is a constant, is that A and B will the same number but different in sign, so let's "guess" something:

$$\frac{1}{x-2} - \frac{1}{x+2}.$$

Hopefully, in your head, you can see that if you combine this into one fraction, then the numerator will be 4. So we will add a "fudge factor".

$$\frac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right).$$

So $A=\frac{3}{4}$ and $B=-\frac{3}{4}$

Example 2: Reduce $\frac{1}{(x-2)(x+5)}$ into partial fractions.

Once again we will guess:

$$\frac{1}{x-2}-\frac{1}{x+5}$$

which will have a numerator of 7 when expanded, so we add the fudge factor.

$$\frac{1}{7}\left(\frac{1}{x-2}-\frac{1}{x+5}\right).$$

As a final remark, it is worth keeping in mind that whenever we have a fraction of the form

$$\frac{c}{(x-\alpha)(x-\beta)},$$

(i.e. a factored *monic* quadratic in the denominator and a *constant* in the numerator) the partial fraction decomposition will be of the form

$$\frac{c}{(x-\alpha)(x-\beta)}\equiv\frac{A}{x-\alpha}+\frac{-A}{x-\beta}.$$

That is, the constants on the numerators in the partial fraction decomposition will *always be negatives of each other.* So once you find one of the constants, you don't have to waste time finding the other, you know automatically it will be the negative of the one you just found.

Example 3: Evaluate the following integral, where g and k are constants.

$$\int\frac{1}{g-kv^2}\,dv$$

This integral is from resisted motion, and the bottom is a difference of two squares.

$$\frac{1}{g-kv^2}=\frac{A}{\sqrt{g}-\sqrt{k}v}+\frac{B}{\sqrt{g}+\sqrt{k}v}$$

Once again, we will guess:

$$\frac{1}{\sqrt{g}-\sqrt{k}v}+\frac{1}{\sqrt{g}+\sqrt{k}v}$$

Notice that this time we guessed them both to be positive, because the denominators are switched around. The numerator is now $2\sqrt{g}$, so we add the fudge factor:

$$
\begin{aligned}
&\frac{1}{2\sqrt{g}}\int \frac{1}{\sqrt{g}-\sqrt{k}v}+\frac{1}{\sqrt{g}+\sqrt{k}v}\,dv \\
&= \frac{1}{2\sqrt{g}}\int \frac{1}{-\sqrt{k}}\frac{-\sqrt{k}}{\sqrt{g}-\sqrt{k}v}+\frac{1}{\sqrt{k}}\frac{\sqrt{k}}{\sqrt{g}+\sqrt{k}v}\,dv \\
&= \frac{1}{2\sqrt{g}}\left(-\frac{1}{\sqrt{k}}\log\left(\sqrt{g}-\sqrt{k}v\right)+\frac{1}{\sqrt{k}}\log\left(\sqrt{g}+\sqrt{k}v\right)\right) \\
&= \frac{1}{2\sqrt{gk}}\log\left(\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right)+C.
\end{aligned}
$$

Heaviside Cover-Up Method

What about expressions like $\frac{2x}{(x+4)(2x-1)}$ where there is an x in the numerator? There is a way to decompose these expressions quickly, called the *Heaviside Cover-Up method*. The method is named after the English electrical engineer, mathematician and physicist Oliver Heaviside.

Example 1: Reduce $\frac{2x}{(x+4)(2x-1)}$ into partial fractions.

We know that this will take the form

$$\frac{A}{x+4}+\frac{B}{2x-1}.$$

In order to find the coefficient of $\frac{1}{x+4}$, we "cover up" the $(x+4)$ factor in the original fraction so we're left with,

$$\frac{2x}{\cancel{(x+4)}(2x-1)}=\frac{2x}{2x-1}$$

and substitute $x=-4$ to obtain A:

$$A=\frac{2(-4)}{2(-4)-1}=\frac{8}{9}.$$

To find the coefficient of $\frac{1}{2x-1}$, we "cover up" the $(2x-1)$ factor in the original fraction so we're left with,

$$\frac{2x}{(x+4)\cancel{(2x-1)}}=\frac{2x}{x+4}$$

and substitute $x=\frac{1}{2}$ to obtain B:

$$B=\frac{2\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)+4}=\frac{2}{9}.$$

So we have successfully decomposed it,

$$\frac{2x}{(x+4)(2x-1)} \equiv \frac{8/9}{x+4} + \frac{2/9}{2x-1}.$$

Example 2: Reduce $\frac{3x-1}{(2-3x)(x+1)}$ into partial fractions.

We know that it will take the form

$$\frac{A}{2-3x} + \frac{B}{x+1}.$$

Applying the Heaviside Cover-Up method, we obtain the coefficients instantly,

$$A = \frac{3\left(\frac{2}{3}\right)-1}{\left(\frac{2}{3}\right)+1} = -\frac{3}{5} \quad ; \quad B = \frac{3(-1)-1}{2-3(-1)} = -\frac{4}{5}.$$

Thus,

$$\frac{3x-1}{(2-3x)(2x-1)} \equiv -\frac{3/5}{2-3x} - \frac{4/5}{x+1}.$$

Example 3: Reduce $\frac{3x^2-2x+1}{(x+1)(x+2)(x+3)}$ into partial fractions.

This method to find partial fractions can be generalised to all linear factors in the denominator. Again, applying the Heaviside Cover-Up method, the coefficients can be obtained very quickly.

We know that the decomposition will take the form of

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Finding the coefficients,

$$A = \frac{3(-1)^2 - 2(-1) + 1}{\big((-1)+2\big)\big((-1)+3\big)} = 3$$

$$B = \frac{3(-2)^2 - 2(-2) + 1}{\big((-2)+1\big)\big((-2)+3\big)} = -17$$

$$C = \frac{3(-3)^2 - 2(-3) + 1}{\big((-3)+1\big)\big((-3)+2\big)} = 17.$$

So the decomposition is

$$\frac{3x^2-2x+1}{(x+1)(x+2)(x+3)} = \frac{3}{x+1} - \frac{17}{x+2} + \frac{17}{x+3}.$$

Using this method, you can even decompose something like

$$\frac{2x^4 - 3x^2 + 8}{(3x-2)(2x+1)(8x-7)(x+1)(5x+3)(4x-9)}.$$

However, we will skip over this and leave it to you as a (tedious) exercise.

6 Advanced Combinatorics

Counting, also called "enumeration" or "combinatorics", can be one of the most demanding and difficult portions of a HSC exam. Probability, a related area, can also be quite challenging.

General tips

- For these problems, *always* set out your working clearly. In most questions, this means listing the cases to be examined, and working out for each case; and then combining these by addition or multiplication to get the final answer.
- Remember to count everything! Don't count anything twice! These are obvious but they are not always easy to put into practice.
- It is useful to spend some time determining what type of problem it is. Does it involve arrangements? Are they ordered? If it's probability, is it binomial? Asking these questions can make it easier to see the 'best' way to do the question.

What type of question is this?

Counting problems (enumeration) in the HSC can be split into three broad categories: ordered selections (permutations), unordered selections (combinations) and ordered arrangements. There have been instances where counting problems were solved as probability ones and vice versa. Make sure you know what type of question you're solving!

Example 1: HSC MX1 1995 Q3a

A security lock has 8 buttons labelled as shown. Each person using the lock is given a 3-letter code.

(i) How many different codes are possible if letters can be repeated and their order is important?

(ii) How many different codes are possible if letters cannot be repeated and their order is important?

(iii) Now suppose that the lock operates by holding 3 buttons down together, so that order is NOT important. How many different codes are possible?

Problem type:

Parts (i) and (ii) are **permutation** problems.

Part (iii) is a **combination** problem.

Example 2: How many ways can the letters of the word INSURABLE be arranged in a row?

Problem type:

This is an **ordered arrangement** question. Order is relevant, and repetitions are not allowed.

Example 3: Consider a lotto style game with a barrel containing twenty similar balls labelled 1 to 20. In each game, four balls are drawn, without replacement, from the twenty balls in the barrel. The probability that any particular number is drawn in the game is 0.2. Find the probability that the number 20 is drawn in exactly two of the next five games played.

Problem type:

This is a **binomial probability** question. What are the values of n? p? q? r?

Example 4: A group of 12 people are to be divided into discussion groups. In how many ways can the discussion groups be formed if there are 3 groups containing 4 people each?

Solution:

1. Select four people from the twelve to go into the first group. There are $\binom{12}{4}$ ways of doing this.
2. Select four people from the remaining eight to go into the second group. There are $\binom{8}{4}$ ways of doing this.
3. Select four people from the remaining four to make the last group. There are $\binom{4}{4}$ ways of doing this.
4. These groups are not different from each other, so we must divide by 3! To avoid counting some ways more than once.

Answer:

$$\frac{\binom{12}{4}\binom{8}{4}\binom{4}{4}}{3!}$$

Now the HSC method looks easy, just writing down the answer. However, for the harder problems you may encounter, it helps to write your logic. This way, silly mistakes are easier to spot as the working is clear and concise.

Let's try a harder example:

Example 5: HSC MX1 2000 Q6

A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and diamonds.

(a) In how many ways can six cards be selected without replacement so that exactly two are spades and four are clubs?
(b) In how many ways can six cards be selected without replacement if at least five cards must be of the same suit?

Solution:

(a)

1. Select two cases for the six card hand. There are $\binom{13}{2}$ ways to do this.
2. Select four clubs. There are $\binom{13}{4}$ ways to do this.

   Answer: $\binom{13}{2}\binom{13}{4}$

(b) We consider two cases:

Case 1: 5 of one suit

1. Select the suit which five cards appear from. There are 4 ways of doing this.
2. Select 5 cards from this suit. There are $\binom{13}{5}$ ways to do this.
3. Select the last card from the remaining 39 cards of the suit not picked. There are $\binom{39}{1}$ ways to do this.

   Subtotal: $4 \times \binom{12}{4} \times \binom{39}{1}$

   Case 2: 6 of one suit

1. Select the suit which appears in the hand. There are 4 ways of doing this.
2. Select six cards from one suit. There are $\binom{13}{6}$ ways to do this.

Subtotal: $4 \times \binom{13}{6}$

Answer: $4 \times \binom{12}{4} \times \binom{39}{1} + 4 \times \binom{13}{6}$

Try these yourself, using the new method:

1. **HSC MX1 2011 Q2c)ii.**
   How many arrangements of the word ALGEBRAIC are possible if the vowels must occupy the 2nd, 3rd, 5th and 8th positions?
2. **HSC MX2 2002 Q4c)ii.**
   From a pack of nine cards numbered 1, 2, 3, …, 9, three cards are drawn at random and placed from left to right. What is the probability that the digits are drawn in descending order? [Hint: for example, 9, 5 and 1 in that order]
3. **HSC MX2 2003 Q4c)i.**
   A hall has n doors. Suppose that n people choose any door at random to enter the hall. In how many ways can this be done?

Finding the best way

Rushing into a question can cause silly mistakes, but this may mean you also fail to recognise a better way to do the question. When given what appears to be a non-standard question, it is worth your while to consider how best to approach the question. Sometimes even using a tree diagram will work out best (we're not kidding).

Example 6: HSC MX2 1996 Q4c)iv.

Let j be an integer from 4 to 20. Show that the probability that, in any one game, j is the largest of the four numbers drawn is $\binom{j-1}{3}/\binom{20}{4}$.

Solution:

The answer is actually simple, but requires knowledge of what is going on in the lotto game, and thus probability and enumeration in general.

What is the total number of possibilities in this lotto game? $\binom{20}{4}$.

We now have to find the number of favourable outcomes. So given an arbitrary integer *j* from 4 to 20, how many favourable outcomes are there? That is, how many times will the other three balls picked have a value less than *j*?

Let us re-state the question as: 'Given an integer *j* from 4 to 20, how many ways can we pick three more numbers that all have a value less than *j*?' We have $j - 1$ numbers to pick from, and we select three of them. Simple. Not convinced? Let us set $j = 4$. So now we want to find out how many times the next three balls will be less than 4. There is only one solution: 1, 2 and 3 – it doesn't matter what order they're picked in. Mathematically, there is $\binom{3}{3} = 1$ ways of doing this selection.

Example 7: HSC MX2 2004 5b)ii. (Modified)*

In how many ways can five **identical** students be placed in three distinct rooms so that no room is empty?

(An interesting) solution:

This uses a special method called "dots, lines and dividers", also known as *stars and bars*.

We have five objects (students), which we want to place in three rooms. We can visualise a solution in a table like this, where • denotes a person.

Room 1	Room 2	Room 3
• •	• •	•
• • •	•	•

The first solution gives two people in rooms 1 and 2, and one in 3. The second solution gives three in room 1, and one each in rooms 2 and 3. We can picture these solutions as a row of dots and lines. For example, the first solution above could be represented as:

• • | • • | •.

So, we have 7 objects – 5 dots and 2 lines – which we must arrange in a row to represent a 'solution'.

Let us pick the places for the 5 dots first. There are $\binom{7}{5}$ ways of doing this. The remaining 2 lines can be put into the remaining 2 places one way. (Check that you get the same answer when you first choose the places for the lines.)

Clearly then, here, the answer is $\binom{7}{5}$. But this is wrong! Why?

Because this is also a solution: • • • • | • |. Reading the question carefully, we also require each room to have at least one person. Or, make sure each line has a dot either side of it. How can we do this using the dots and lines method? We instead think of it this way.

We have the five dots, • • • • •. Ensuring that there is at least one person in each room is the same as ensuring that a line is always besides two dots (i.e. • | • kind of arrangement). We think about the number of ways to place our two lines between the dots, with the possible positions underscored, • _ • _ • _ • _ •. This guarantees that each room will have at least one student in it. So we have 4 spots (the underscores), 2 lines and selecting two of these spots for the lines, our answer is $\binom{4}{2} = 6$.

*****The question differs from the original since here the students are identical, while in the original they are distinguishable. This question was modified to outline the stars and bars technique.

7 Harder Inequalities

Often, the more difficult questions in the Ext 2 paper deal with inequality proofs. These proofs fall into two categories: algebraic proofs dealing with things like

$$\frac{a+b}{2} \geq \sqrt{ab}$$

or calculus proofs dealing with inequalities of functions and their limits, derivatives, integrals, as well as graphical arguments.

General tips

- The result you are required to prove may look challenging but working backwards can make the problem much simpler to unpack. You should always work backwards on scrap paper, then write the solution in the correct order.
- Make use of anything given in the question. If the question supposes an initial inequality and asks to prove another inequality, your first line in your solution should be the initial inequality.
- Identify whether the inequality is an algebraic one or a calculus one. This will tell you whether the techniques you need to use are things like AM-GM or derivatives/integrals.
- Be careful of sign changes when multiplying! This is an easy mistake to make and hard to catch in the moment. Also make sure you aren't dividing by a possible zero!

Useful techniques for your toolbox:

1. The Arithmetic Mean – Geometric Mean Inequality (AM-GM): For any n non-negative real numbers $x_1, x_2, \dots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}.$$

The most common use of this inequality is the $n = 2$ case, where

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}.$$

2. Pinching Theorem: If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, then $\lim_{x\to\infty} f(x) = L$. Intuitively speaking, if a function f is between two other functions which approach the same value when x approaches ∞, then our function f will also approach this value. (Also known as: Sandwich theorem, Squeeze theorem)
3. Rate of change inequalities: If $f(a) \geq g(a)$ and $f'(x) \geq g'(x)$ for all $x \geq a$, then $f(x) \geq g(x)$ for all $x \geq a$. Intuitively speaking, if two functions are at the same place at some point a, and one of the functions is moving faster, than the faster function will be greater than the slower function for all subsequent points after a.

Example 1: HSC MX2 2018 15c).

Let n be a positive integer and let x be a positive real number.

(i) Show that $x^n - 1 - n(x-1) = (x-1)(1 + x + \cdots + x^{n-1} - n)$.

(ii) Hence show that $x^n \geq 1 + n(x-1)$.

(iii) Deduce that for positive real numbers a and b, $a^n b^{1-n} \geq na + (1-n)b$.

Solution:

First recall that $x^n - 1 = (x-1)(1 + x + \cdots + x^{n-1})$. Then

$$x^n - 1 - n(x-1) = (x-1)(1 + x + \cdots + x^{n-1}) - n(x-1)$$

$$x^n - 1 - n(x-1) = (x-1)(1 + x + \cdots + x^{n-1} - n).$$

For part 2, notice that what we are required to show is equivalent to

$$x^n - 1 - n(x-1) \geq 0.$$

From part 1,

$$x^n - 1 - n(x-1) = (x-1)(1 + x + \cdots + x^{n-1} - n)$$

so we want to show that $(x-1)(1 + x + \cdots + x^{n-1} - n) \geq 0$. Since there is a zero point at $x = 1$ we are motivated to do this by cases. For $0 < x < 1$,

$$x - 1 < 0 \qquad \text{and} \qquad x^k < 1. \qquad\qquad (k \geq 1)$$

Hence $x^1 + \cdots + x^{n-1} < n - 1$, i.e.

$$1 + x + \cdots + x^{n-1} - n < 0 .$$

So $(x-1)(1 + x + \cdots + x^{n-1} - n) > 0$ since each part of the LHS are less than zero.

For $x = 1$,

$$x^n - 1 - n(x-1) = 1 - 1 - n(1-1) = 0 \geq 0 .$$

For $x > 1$,

$$x - 1 > 0 \qquad \text{and} \qquad x^k > 1. \qquad\qquad (k \geq 1)$$

Hence $x^1 + \cdots + x^{n-1} > n - 1$, i.e.

$$1 + x + \cdots + x^{n-1} - n > 0 .$$

So $(x-1)(1 + x + \cdots + x^{n-1} - n) > 0$ since each part of the LHS are greater than zero.

Therefore, by cases,

$$(x-1)(1+x+\cdots+x^{n-1}-n) \geq 0,$$

i.e.
$$x^n \geq 1+n(x-1)\ .$$

Now for part 3, let $x = ab^{-1}$. Then

$$a^n b^{-n} \geq 1+n(ab^{-1}-1)$$

$$a^n b^{1-n} \geq b+n(a-b)$$

$$a^n b^{1-n} \geq na+(1-n)b.$$

Example 2: HSC MX2 2015 15c).

For positive real numbers x and y , $\sqrt{xy} \leq \frac{x+y}{2}$.

(i) Prove $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$, for positive real numbers x and y .

(ii) Prove $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$, for positive real numbers a, b, c and d .

First notice that the identity we are given is the AM-GM inequality for $n = 2$. From here we can substitute x^2 for x and y^2 for y to get:

$$\sqrt{x^2y^2} \leq \frac{x^2+y^2}{2}$$

$$\mathrm{xy} \leq \frac{x^2+y^2}{2}$$

$$\sqrt{\mathrm{xy}} \leq \sqrt{\frac{x^2+y^2}{2}}.$$

Now for part 2, we need to use the proven inequality from part 1. For positive real numbers a, b, c and d, we have

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}} \quad and \quad \sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$$

Multiplying the inequalities, we get

$$\sqrt{abcd} \leq \sqrt{\frac{a^2+b^2}{2} \cdot \frac{c^2+d^2}{2}}\ .$$

However, we can apply the AM-GM inequality to the RHS so that

$$\sqrt{\frac{a^2+b^2}{2}\cdot\frac{c^2+d^2}{2}} \le \frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2}$$

$$\sqrt{\frac{a^2+b^2}{2}\cdot\frac{c^2+d^2}{2}} \le \frac{a^2+b^2+c^2+d^2}{4}\ .$$

Hence

$$\sqrt{abcd} \le \frac{a^2+b^2+c^2+d^2}{4}.$$

Here we simply take square roots of both sides to arrive at the required inequality:

$$\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

Example 3: HSC MX2 2014 16 b).

Suppose n is a positive integer.

(i) Show that

$$-x^{2n} \le \frac{1}{1+x^2} - (1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}) \le x^{2n}\ .$$

(ii) Use integration to deduce that

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}.$$

(iii) Explain why

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Solution:

First, note that if we multiply the desired inequality by $1+x^2$, then our middle term becomes

$$\begin{aligned}1 - (1+x^2)(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}) &= 1 - (1 - (-x^2)^n)\\ &= (-x^2)^n.\end{aligned}$$

Hence we have

$$\frac{1}{1+x^2} - (1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}) = \frac{(-x^2)^n}{1+x^2}.$$

Taking absolute values,

$$\left|\frac{(-x^2)^n}{1+x^2}\right| = \frac{|-x^2|^n}{|1+x^2|} = \frac{x^{2n}}{1+x^2} \le x^{2n} \,,$$

so then by unpackaging the absolute value:

$$-x^{2n} \le \frac{(-x^2)^n}{1+x^2} \le x^{2n} \,.$$

Hence

$$-x^{2n} \le \frac{1}{1+x^2} - (1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}) \le x^{2n} \,.$$

Now we will integrate our inequality from $x = 0$ to 1:

$$\int_0^1 -x^{2n}dx = -\frac{1}{2n+1},$$

$$\int_0^1 \frac{1}{1+x^2} - (1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2})\,dx$$
$$= \tan^{-1}1 - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right),$$

and

$$\int_0^1 x^{2n}dx = \frac{1}{2n+1}.$$

$\tan^{-1}1 = \frac{\pi}{4}$, so

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}.$$

Now we can see that we have a pinching theorem situation. By taking the limit as $n \to \infty$ we see that

$$\lim_{n\to\infty} -\frac{1}{2n+1} = \lim_{n\to\infty}\frac{1}{2n+1} = 0\,.$$

So

$$0 \le \lim_{n\to\infty}\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) \le 0.$$

Then by the pinching theorem we have

$$\lim_{n\to\infty} \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) = 0$$

$$\frac{\pi}{4} - \lim_{n\to\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) = 0\,.$$

Hence

$$\frac{\pi}{4} = \lim_{n\to\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$